Review Questions 1

Solutions

1. Compute the differentials of the functions below.

a. $y = x^2 - 3x + 1, \quad dy = (2x - 3)\,dx$

b. $u = e^{x^2-3x+1}, \quad du = e^{x^2-3x+1}(2x - 3)\,du$

2. Use differentials to estimate $\sqrt[3]{28}$. Express your answer as a simple fraction, a/b, **not** in decimal form.

We use the approximation formula $f(x_0 + dx) \approx f(x_0) + dy$. First, we identify the function, which is straightforward: $f(x) = x^{1/3}$. Next, we identify x_0. We want to set $28 = x_0 + dx$, and we want dx to be relatively small, and we also want x_0 to be a point for which it is easy to evaluate $f(x)$. In other words we are looking for a point, x_0, that is close to 28 and for which the cube root is known. $x_0 = 27$ fills the bill.

So, we have $f(x) = x^{1/3}$, $x_0 = 27$ and $dx = 28 - 27 = 1$. Next we compute dy:

$$dy = f'(x_0)dx = \frac{1}{3}x_0^{-2/3}dx = \frac{1}{3}27^{-2/3} \cdot 1 = \frac{1}{27}.$$

Finally, we plug everything back into the approximation formula, above.

$$28^{1/3} \approx 27^{1/3} + dy = 3 + \frac{1}{27} = \frac{82}{27}.$$

Note: estimate is within 0.00045 of the true value of $\sqrt[3]{28}$.

3. Compute the indefinite integrals below.

a. $\displaystyle\int 3x^4 - 2x^3 + 6x^2 + 2x - 1\,dx = \frac{3}{5}x^5 - \frac{1}{2}x^4 + 2x^3 + x^2 - x + C.$

b. $\displaystyle\int \sqrt[5]{x^3}\,dx = \int x^{3/5}\,dx = \frac{5}{8}x^{8/5} + C.$

c. $\displaystyle\int \frac{3x^2 - 4x + 1}{x^5}\,dx = \int 3x^{-3} - 4x^{-4} + x^{-5}\,dx = -\frac{3}{2}x^{-2} + \frac{4}{3}x^{-3} - \frac{1}{4}x^{-4} + C.$

4. Find the function $y = f(x)$, given that $y' = x - \dfrac{1}{x}$, and $f(1) = 3$.

First, we integrate

$$\int y'\,dx = \int x - \frac{1}{x}\,dx = \frac{x^2}{2} - \ln|x| + C.$$

This means that $f(x) = \frac{x^2}{2} - \ln|x| + C$, and we use the initial value to solve for C.

$$3 = f(1) = \frac{1^2}{2} - \ln 1 + C = \frac{1}{2} + C \implies C = 3 - \frac{1}{2} = \frac{5}{2}.$$

So, $f(x) = \dfrac{x^2}{2} - \ln|x| + \dfrac{5}{2}$.

5. Find the function $y = g(x)$, given that $y'' = x^2 - 1$, $y'(1) = 2$ and $y(1) = 2$.

First, we solve one initial value problem to find y', and to do this we begin by integrating y''.

$$y' = \int y''\,dx = \int x^2 - 1\,dx = \frac{x^3}{3} - x + C_1.$$

Next, we solve for C_1 using the initial value for y',

$$2 = y'(1) = \frac{1^3}{3} - 1 + C_1 \implies C_1 = 2 + 1 - \frac{1}{3} = \frac{8}{3}.$$

So, $y' = \frac{x^3}{3} - x + \frac{8}{3}$, and now we repeat the process to find $y = g(x)$.

$$g(x) = \int y'\,dx = \int \frac{x^3}{3} - x + \frac{8}{3}\,dx = \frac{x^4}{12} - \frac{x^2}{2} + \frac{8x}{3} + C_2.$$

Finally, use the initial data for $g(x)$ to solve for C_2,

$$2 = g(1) = \frac{1}{12} - \frac{1}{2} + \frac{8}{3} + C_2 \implies C_2 = -\frac{1}{4},$$

giving the final solution $g(x) = \dfrac{x^4}{12} - \dfrac{x^2}{2} + \dfrac{8x}{3} - \dfrac{1}{4}$.

6. The demand function $p = f(q)$ is found by dividing the revenue function, $r(q)$, by q, i.e., $p = r/q$. The revenue function is found by solving the initial value problem, $r' = 200 - q^{2/3}, \quad r(0) = 0$. First,

$$r = \int 200 - q^{2/3}\,dq = 200q - \frac{3}{5}q^{5/3} + C.$$

Next, the initial value $r(0) = 0$ implies that $C = 0$, so $r = 200q - \frac{3}{5}q^{5/3}$, and the demand function is

$$p = 200 - \frac{3}{5}q^{2/3}.$$

7. Another initial value problem. First,

$$c = \int (q + 1000)^{1/3} + 50\, dq = \frac{3}{4}(1000 + q)^{4/3} + 50q + K,$$

(using the substitution $u = 1000 + q$ to find the integral). Next, the initial data is $c(0) = 12000$ (fixed cost $= c(0)$), which we use to solve for the constant of integration, K:

$$12000 = c(0) = \frac{3}{4}1000^{4/3} + K = 7500 + K \implies K = 4500.$$

Thus, the cost function is $c = \dfrac{3}{4}(1000 + q)^{4/3} + 50q + 4500.$