Review Questions 2

Solutions

1. Compute the following integrals

a. Substitute $u = x^2 + 1$, $du = 2x\,dx$, then $3x\,dx = (3/2)du$ and

$$\int \frac{3x\,dx}{\sqrt[3]{x^2+1}} = \frac{3}{2}\int u^{-1/3}\,du = \frac{3}{2}\cdot\frac{u^{2/3}}{2/3} + C = \frac{9}{4}(x^2+1)^{2/3} + C.$$

b. Substitute $u = x^3 + 3x^2 - 1$, $du = (3x^2 + 6x)\,dx$, then $(x^2 + 2x)\,dx = (1/3)\,du$ and

$$\int (x^2+2x)(x^3+3x^2-1)^3\,dx = \frac{1}{3}\int u^3\,du = \frac{1}{12}u^4 + C = \frac{1}{12}(x^3+3x^2-1)^4 + C.$$

c. Substitute $v = \ln x$, $dv = \frac{1}{x}\,dx$, then

$$\int \frac{dx}{x\ln x} = \int \frac{1}{\ln x}\cdot\frac{1}{x}\,dx = \int \frac{dv}{v} = \ln|v| + C = \ln|\ln x| + C.$$

d. Substitution does ***not*** work here (why?), but the integrand is a polynomial:

$$\int (x^2+x)(x^3+x^2-1)^2\,dx = \int (x^2+x)(x^6+2x^5+x^4-2x^3-2x^2+1)\,dx =$$
$$\int x^8 + 2x^7 + 3x^6 - x^5 - 4x^4 - 2x^3 + x^2 + x\,dx = \frac{x^9}{9} + \frac{x^8}{4} + \frac{3x^7}{7} - \frac{x^6}{6} - \frac{4x^5}{5} -$$
$$\frac{x^4}{2} + \frac{x^3}{3} + \frac{x^2}{2} + C.$$

e. Substitute $u = \sqrt{x^2+1}$, giving $du = \dfrac{x}{\sqrt{x^2+1}}\,dx$, (check!!), so

$$\int \frac{3x\cdot e^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}\,dx = 3\int e^u\,du = 3e^u + C = 3e^{\sqrt{x^2+1}} + C.$$

Sometimes the 'ugliest' integrals are very easy.

f. Substitute $u = -0.05t$, $du = -.05\,dt$, then $dt = -20\,du$ and

$$\int 1000e^{-0.05t}\,dt = -20000\int e^u\,du = -20000e^u + C = -20000e^-0.05t + C.$$

g. Substitute $u = 3t + 1$, $du = 3\,dt$, so $dt = \frac{1}{3}\,du$ and $t = \frac{1}{3}(u-1)$. Then

$$\int \frac{t^2+5}{3t+1}\,dt = \frac{1}{3}\int \frac{\frac{1}{9}(u-1)^2+5}{u}\,du = \frac{1}{3}\int \frac{1}{9}u - \frac{2}{9} + \frac{46}{9}\frac{1}{u}\,du$$
$$= \frac{1}{27}\left(\frac{u^2}{2} - 2u + 46\ln|u|\right) + C = \frac{(3t+1)^2}{54} - \frac{2(3t+1)}{27} + \frac{46}{27}\ln|3t+1| + C$$
$$= \frac{t^2}{6} - \frac{t}{9} + \frac{46}{27}\ln|3t+1| + C.$$ (Constants have been absorbed by C.)

h. Substitute $u = 2x+1$, $du = 2\,dx$. This implies that $dx = \frac{1}{2}du$ and $x = \frac{u-1}{2}$:

$$\int \frac{2+x}{2x+1}\,dx = \frac{1}{2}\int \frac{2+\frac{u-1}{2}}{u}\,du = \frac{1}{4}\int \frac{u+3}{u}\,du = \frac{1}{4}\int 1 + \frac{3}{u}\,du$$
$$= \frac{u}{4} + \frac{3}{4}\ln|u| + C = \frac{2x+1}{4} + \frac{3}{4}\ln|2x+1| + C = \frac{x}{2} + \frac{3}{4}\ln|2x+1| + C.$$

How did $\frac{2x+1}{4}$ become $\frac{x}{2}$? In other words, what happened to the $\frac{1}{4}$?

2. First, integrate

$$c = \int 2(5q+100)^{1/2}\,dq = \frac{4}{15}(5q+100)^{3/2} + K,$$

(using the substitution $u = 5q+100$, $du = 5\,dq$, so $dq = \frac{1}{5}\,du$). Next, use the initial data, namely $c(0) = \text{fixed cost} = 10000$, to solve for K.

$$10000 = c(0) = \frac{4}{15}(0+100)^{3/2} + K = \frac{4000}{15} + K \implies K = \frac{146000}{15},$$

and the cost function is $c = \dfrac{4(5q+100)^{3/2} + 146000}{15}$.

3. The firm's profit is the difference between their revenue and their cost, $\pi = r - c$, so the change in the firm's profit is the difference between the change in their revenue and the change in their cost. I.e.,

$$\pi(200)-\pi(100) = [r(200)-c(200)]-[r(100)-c(100)] = [r(200)-r(100)]-[c(200)-c(100)].$$

To find the changes in revenue and cost, we use the idea discussed in class. Namely, if

$$r = f(q) + C_1 \text{ and } c = g(q) + C_2,$$

then

$$\begin{aligned} r(200) - r(100) &= f(200) + C_1 - [f(100) + C_1] \\ &= f(200) - f(100) + C_1 - C_1 \\ &= f(200) - f(100), \end{aligned}$$

and

$$\begin{aligned} c(200) - c(100) &= g(200) + C_2 - [g(100) + C_2] \\ &= g(200) - g(100) + C_2 - C_2 \\ &= g(200) - g(100). \end{aligned}$$

In other words, to find the *total change* in the value of r, for example, we can use *any* antiderivative of dr/dq — the constant of integration does not play a role in this problem.

To find the change in the value of the revenue function, we first find the antiderivatives of dr/dq, (using the substitution $u = 2q+8$, $dq = du/2$):

$$\int 200 - (2q+8)^{2/3}\,dq = 200q - \frac{1}{2}\cdot\frac{(2q+8)^{5/3}}{5/3} + C_1 = 200q - 0.3(2q+8)^{5/3} + C_1.$$

So $r = 200q - 0.3(2q+8)^{5/3} + C_1$, for some constant C_1, and therefore

$$\begin{aligned} r(200) - r(100) &= \left(200\cdot 200 - (0.3)\cdot(408)^{5/3} + C_1\right) - \left(200\cdot 100 - (0.3)\cdot(208)^{5/3} + C_1\right) \\ &\approx 33266.80 - 17809.41 \\ &= 15457.39. \end{aligned}$$

Now we repeat this procedure for the cost function:

$$\int 0.2q + 65\,dx = 0.1q^2 + 65q + C_2,$$

so $c = 0.1q^2 + 65q + C_2$ and the change in the value of the cost function is

$$c(200) - c(100) = [0.1\cdot 200^2 + 65\cdot 200 + C_1] - [0.1\cdot 100^2 + 65\cdot 100 + C_1] = 17000 - 7500 = 9500.$$

Finally, the change in profit is $15457.39 - 9500 = 5957.39$.

4. First, we integrate the marginal revenue function:

$$\begin{aligned} \int \frac{dr}{dq}\,dq &= \int 50 - \frac{(\ln(q+1)+1)^5}{q+1}\,dq \\ &= \int 50\,dq - \int \frac{(\ln(q+1)+1)^5}{q+1}\,dq \\ &= 50q - \int u^5\,du \qquad \left(\text{substitute } u = \ln(q+1)+1, \text{ so } du = \frac{1}{q+1}\,dq.\right) \\ &= 50q - \frac{u^6}{6} + C \\ &= 50q - \frac{(\ln(q+1)+1)^6}{6} + C \\ \text{so, } r &= 50q - \frac{(\ln(q+1)+1)^6}{6} + C. \end{aligned}$$

Now, use the initial value $r(0) = 0$ to solve for the constant of integration C;

$$0 = r(0) = 50\cdot 0 - \frac{(\ln(1)+1)^6}{6} + C = -\frac{1}{6} + C,$$

which means that $C = 1/6$ and

$$r = 50q - \frac{(\ln(q+1)+1)^6}{6} + \frac{1}{6}.$$

5. Suppose that a small nation's marginal propensity to consume is given by

$$\frac{dC}{dY} = \frac{63Y^2 + 70Y - 450}{(9Y+5)^2},$$

where Y is income and C is consumption, both measured in billions of dollars.

a. $\displaystyle\lim_{Y\to\infty} \frac{dC}{dY} = \lim_{Y\to\infty} \frac{63Y^2 + 70Y - 450}{(9Y+5)^2} = \lim_{Y\to\infty} \frac{63Y^2 + 70Y - 450}{81Y^2 + 90Y + 25} = \frac{63}{81} = \frac{7}{9}.$
Interpretation: When income is very large the nation consumes approximately 7/9 of each additional dollar of income.

b. This is an initial value problem. First we integrate, using the substitution $u = 9Y + 5$, which means that $dY = \frac{1}{9}du$ and $Y = \frac{u-5}{9}$:

$$\begin{aligned}
\int \frac{63Y^2 + 70Y - 450}{(9Y+5)^2}\, dY &= \frac{1}{9}\int \frac{63\left(\frac{u-5}{9}\right)^2 + 70\left(\frac{u-5}{9}\right) - 450}{u^2}\, du \\
&= \frac{1}{9}\int \frac{\frac{63}{81}(u^2 - 10u + 25) + \frac{70}{9}(u-5) - 450}{u^2}\, du \\
&= \frac{1}{9}\int \frac{\frac{7}{9}u^2 - \frac{4225}{9}}{u^2}\, du \\
&= \frac{7}{81}\int du - \frac{4225}{81}\int u^{-2}\, du \\
&= \frac{7u}{81} + \frac{4225}{81u} + K \\
&= \frac{7(9Y+5)}{81} + \frac{4225}{81(9Y+5)} + K \\
&= \frac{7Y}{9} + \frac{4225}{729Y + 405} + K.
\end{aligned}$$

So, $C = \dfrac{7Y}{9} + \dfrac{4225}{729Y + 405} + K$, where in this example, K is the constant of integration. To solve for K we use the given data: $C(5) = 4.5$.

$$4.5 = C(5) = \frac{35}{9} + \frac{4225}{4050} + K = \frac{799}{162} + K \implies \boxed{K = -\frac{35}{81}.}$$

Thus the consumption function is

$$C = \frac{7Y}{9} + \frac{4225}{729Y + 405} - \frac{35}{81} = \frac{7Y^2 + 50}{9Y + 5}.$$