Review Questions 3

Solutions

1. Compute the following **definite** integrals. Use the *Fundamental Theorem of Calculus*, (i.e. don't use limits of right-hand sums).

a. $\displaystyle\int_0^4 \frac{3x\,dx}{\sqrt[3]{x^2+1}} = \frac{9}{4}(x^2+1)^{2/3}\Big|_0^4 = \frac{9}{4}\left(17^{2/3} - 1^{2/3}\right) \approx 12.62585.$

b. $\displaystyle\int_1^3 (x^2+2x)(x^3+3x^2-1)^3\,dx = \frac{1}{12}(x^3+3x^2-1)^4\Big|_1^3 = \frac{1}{12}\left(53^4 - 3^4\right) = 657533.\overline{333}.$

c. $\displaystyle\int_e^{e^2} \frac{dx}{x\ln x} = \ln|\ln x|\Big|_e^{e^2} = \ln(\ln e^2) - \ln(\ln e) = \ln 2 - \ln 1 = \ln 2.$

Note: Problems a., b. and c., above, involve the same integrals that appeared in 1abc of RQ 5. See the solutions of RQ 5 to see how the antiderivatives were found.

d. Substitute $u = -0.05t$, $du = -0.05\,dt \Rightarrow dt = -20\,du$, and also, change the limits of integration according to the substitution: $t = 0 \Rightarrow u = -(0.05)\cdot 0 = 0$ and $t = 10 \Rightarrow u = -(0.05)\cdot 10 = -0.5$. Then,

$$\int_0^{10} 1000e^{-0.05t}\,dt = \int_0^{-0.5} -20000e^u\,du = -20000e^u\Big|_0^{-0.5} = -20000\left(e^{-0.5} - e^0\right) \approx 7869.387.$$

e. Substitute $u = x^2+1$, $du = 2x\,dx$, so $x\,dx = 0.5du$. Also, $x = 0 \Rightarrow u = 1$ and $x = 1 \Rightarrow u = 2$, so

$$\int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_1^2 \frac{du}{u} = \frac{\ln u}{2}\Big|_1^2 = \frac{\ln 2 - \ln 1}{2} = \frac{\ln 2}{2}.$$

f. Substitute $v = e^x + e^{-x}$, then $dv = (e^x - e^{-x})\,dx$, and

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{dv}{v} = \ln v + C = \ln(e^x + e^{-x}) + C.$$

This gives

$$\int_0^2 \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \ln(e^x + e^{-x})\Big|_0^2 = \ln(e^2+e^{-2}) - \ln(e^0+e^0) = \ln\left(\frac{e^2+e^{-2}}{2}\right)\ (\approx 1.325).$$

Note: In problems d. and e. I substituted directly in the definite integral, and the limits of integration changed accordingly. In f., I first computed the indefinite integral of the integrand, and then used this to compute the definite integral in terms of the original variable of integration, so there was no need to change the limits of integration. This was also the case in a.,b. and c., above.

Conclusion: If you change the variable of integration in a **definite** integral (via substitution), then the limits of integration must change accordingly.

2. Compute ...

a. First note that $\sum_{k=10}^{20} 2k+1 = \sum_{k=1}^{20} 2k+1 - \sum_{k=1}^{9} 2k+1$.

Next, note that $\sum_{k=1}^{n} 2k+1 = 2\left(\sum_{k=1}^{n} k\right) + \left(\sum_{k=1}^{n} 1\right) = n(n+1)+n = n^2+2n$, as we see by using the formulas

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^{n} 1 = n.$$

This means that $\sum_{k=10}^{20} 2k+1 = (20^2+40) - (9^2+18) = 341$.

b. $\sum_{i=0}^{100} i^2+3i-4 = \sum_{i=1}^{100} i^2 + \sum_{i=1}^{100} 3i - \sum_{i=0}^{100} 4 = \sum_{i=1}^{100} i^2 + 3\sum_{i=1}^{100} i - 4\sum_{i=0}^{100} 1 =$

$= \dfrac{100\cdot 101\cdot 201}{6} + 3\left(\dfrac{100\cdot 101}{2}\right) - 404 = 353096.$

I split the sum (again), pulled out constant factors (again), and in addition to the summation formulæ that I used in a., I also used

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

The only other point of interest here is that when $i=0$, the terms i^2 and $3i$ are also 0, so I omitted these terms from the corresponding sums on the right (but I didn't do this for the constant term 4).

c. In problem **7.** from the exercises in SN 1, you were asked to show that

$$\sum_{k=1}^{n} kq^k = \frac{nq^{n+1}}{q-1} - \frac{q^{n+1}-q}{(q-1)^2}.$$

In this problem, all you have to do is *use* the formula. To do this, set $q = e^{-0.05}$, then

$$\sum_{m=1}^{100} me^{-0.05m} = \sum_{m=1}^{100} m\left(e^{-0.05}\right)^m = \frac{100e^{-5.1}}{e^{-0.05}-1} - \frac{e^{-5.05}-e^{-0.05}}{(e^{-0.05}-1)^2} \approx 384.7212$$

3. The change in consumption is given by

$$\Delta C = \int_{10}^{15} \frac{dC}{dY}\, dY = \int_{10}^{15} \frac{8Y+13}{9Y+21}\, dY.$$

To compute this integral, we make the substitution $u = 9Y+21$. This entails the following changes

- $du = 9dY \implies dY = \frac{1}{9}du.$
- $Y = \frac{u-21}{9}.$
- And the limits of integration change: $Y = 10 \implies u = 111$ and $Y = 15 \implies u = 156.$

Thus, the change in consumption is ...

$$\begin{aligned} \Delta C = \int_{10}^{15} \frac{8Y+13}{9Y+21}\, dY &= \frac{1}{9}\int_{111}^{156} \frac{8(u-21)/9+13}{u}\, du \\ &= \frac{1}{9}\int_{111}^{156} \frac{\frac{8}{9}u - \frac{17}{3}}{u}\, du \\ &= \frac{1}{9}\int_{111}^{156} \frac{8}{9} - \frac{17}{3}\cdot\frac{1}{u}\, du \\ &= \frac{8}{81}u - \frac{17}{27}\ln|u|\Big|_{111}^{156} = \\ &= \left(\frac{1248}{81} - \frac{17}{27}\ln 156\right) - \left(\frac{888}{81} - \frac{17}{27}\ln 111\right) \approx 4.23. \end{aligned}$$

When income increases from \$10 billion to \$15 billion, consumption increases by about \$4.23 billion and consequently, savings increase by \$0.77 billion. ($S = Y - C$)

4. A firm's marginal revenue function is $\frac{dr}{dq} = q\sqrt{1000-0.1q^2}$. Suppose that the firm's output increases from $q = 40$ to $q = 50$. Then the total change in revenue is

$$\Delta r = \int_{40}^{50} \frac{dr}{dq}\, dq = \int_{40}^{50} q\sqrt{1000-0.1q^2}\, dq$$

To compute the integral, substitute: $u = 1000 - 0.1q^2,\ du = -0.2q\, dq \Rightarrow q\, dq = -5\, du,$

and note how the limits of integration change...

$$\Delta r = \int_{40}^{50} q\sqrt{1000 - 0.1q^2}\, dq = -5\int_{840}^{750} u^{1/2}\, du$$
$$= -5 \cdot \frac{u^{3/2}}{3/2}\bigg|_{840}^{750} = -\frac{10}{3}\left(750^{3/2} - 840^{3/2}\right) \approx 12686.39.$$