Review Questions 4

Solutions

Note: The formula numbers below refer to the table of integral formulas in Appendix C.

1. a. $\displaystyle\int \frac{4\,dx}{5x\sqrt{x^2+9}} = \frac{4}{5}\left(\frac{1}{3}\ln\left|\frac{\sqrt{x^2+9}-3}{x}\right| + C\right) = \frac{4}{15}\ln\left|\frac{\sqrt{x^2+9}-3}{x}\right| + C,$

(Formula #28, with $a = 3$).

b. $\displaystyle\int_0^4 \frac{2x\,dx}{\sqrt{9+4x}} = 2\left(\frac{2(4x-18)\sqrt{9+4x}}{48}\right)\Bigg|_0^4 = \frac{4\cdot(-2)\cdot\sqrt{25}}{48} - \frac{4\cdot(-18)\sqrt{9}}{48} = \frac{11}{3},$

(Formula #15, with $a = 9$ and $b = 4$).

c. First, using formula #39 (with $n = 2$), followed by formula #38, (and simplifying), we have

$$\begin{aligned}
\int t^2 e^{-0.06t}\,dt &= \frac{t^2 e^{-0.06t}}{-0.06} - \frac{2}{-0.06}\int t e^{-0.06t}\,dt \\
&= \frac{t^2 e^{-0.06t}}{-0.06} + \frac{2}{0.06}\left(\frac{e^{-0.06t}}{(0.06)^2}(-0.06t-1) + C\right) \\
&= -\frac{e^{-0.06t}}{(0.06)^3}\left((0.06)^2t^2 + (0.12)t + 2\right) + C.
\end{aligned}$$

So,

$$\begin{aligned}
\int_0^{10} 200t^2 e^{-0.06t}\,dt &= -\frac{200e^{-0.06t}}{(0.06)^3}\left((0.06)^2t^2 + (0.12)t + 2\right)\Bigg|_0^{10} \\
&= -\frac{200e^{-0.6}}{(0.06)^3}(0.36 + 1.2 + 2) - \left(-\frac{200}{(0.06)^3}\cdot 2\right) \approx 42806.09.
\end{aligned}$$

d. $\displaystyle\int \frac{3\,dv}{\sqrt{4v^2+25}} = \frac{3}{2}\int \frac{du}{\sqrt{u^2+25}} = \frac{3}{2}\ln\left|u + \sqrt{u^2+25}\right| + C$

$\displaystyle = \frac{3}{2}\ln\left|2v + \sqrt{4v^2+25}\right| + C,$

(substitute $u = 2v$, so $4v^2 = u^2$ and $dv = \frac{1}{2}du$, then use formula #27 with $a = 5$).

e. $\displaystyle\int 5x^3 \ln x\,dx = \frac{5x^4 \ln x}{4} - \frac{5x^4}{16} + C,$

(use formula #42 with $n = 3$).

f. You can compute this integral by splitting the numerator, and computing two integrals using formulas #2 and #3, or you can make a substitution, as follows. Substitute $u = 2+7x$, then $x = (u-2)/7$, so $3+5x = 3+5(u-2)/7 = (11 + 5u)/7$, and $dx = \frac{1}{7}du$. Remember that the limits of integration also change, i.e., $x = 0 \Rightarrow u = 2$ and $x = 2 \Rightarrow u = 16$. Putting all of this together gives

$$\begin{aligned} \int_0^2 \frac{3+5x}{2+7x}\,dx &= \frac{1}{49}\int_2^{16} \frac{11+5u}{u}\,du \\ &= \frac{1}{49}(11\ln u + 5u)\Big|_2^{16} \\ &= \frac{11\ln 8 + 70}{49} \qquad (\approx 1.89538483589). \end{aligned}$$

Note: When I leave out steps (as above), you should fill them in on your own, to make sure you understand what's going on.

2. First, find the equilibrium price and quantity, by setting demand equal to supply:

$$p = 0.1q+5 = 40-\frac{q}{10}-\frac{q^2}{100} \implies 0.01q^2+0.2q-35 = 0 \implies q = 50 \text{ or } q = -70.$$

Since quantity must be positive, the equilibrium quantity is $\tilde{q} = 50$, and the equilibrium price is $\tilde{p} = 0.1\tilde{q}+5 = 10$. Now compute the Consumers' Surplus and Producers' Surplus.

$$\begin{aligned} \text{Consumers' Surplus} &= \int_0^{\tilde{q}} (\text{demand} - \tilde{p})\,dq \\ &= \int_0^{50} 40 - \frac{q}{10} - \frac{q^2}{100} - 10\,dq \\ &= 30q - \frac{q^2}{20} - \frac{q^3}{300}\Big|_0^{50} = \frac{2875}{3} \approx 958.33. \\ \text{Producers' Surplus} &= \int_0^{\tilde{q}} (\tilde{p} - \text{supply})\,dq \\ &= \int_0^{50} 10 - (0.1q+5)\,dq \\ &= 5q - \frac{q^2}{20}\Big|_0^{50} = 125. \end{aligned}$$

3. Gini coefficient, γ:

$$\gamma = 1 - 2\int_0^1 0.3x^3 + 0.2x^2 + 0.5x\,dx = 1 - 2\left(\frac{0.3x^4}{4} + \frac{0.2x^3}{3} + \frac{0.5x^2}{2}\bigg|_0^1\right) = \frac{13}{60}.$$

4. See section 15.3 in the text.

$$\begin{aligned}\text{Present Value} &= \int_0^T f(t)e^{-rt}\,dt \\ &= \int_0^{20} 250te^{-0.0475t}\,dt \\ &= \frac{250e^{-0.0475t}}{(0.0475)^2}(-0.0475t - 1)\bigg|_0^{20} \approx 27241.55.\end{aligned}$$

5. The average value of the cost function, $c = 0.05q^2 + 35q + 12000$, on the interval $[0, 100]$ is

$$\text{Avg}(c) = \frac{1}{100}\int_0^{100} 0.05q^2 + 35q + 12000\,dq = \frac{1}{100}\left(\frac{0.05q^3}{3} + \frac{35q^2}{2} + 12000q\right)\bigg|_0^{100} \approx 13916.666.$$

The value of the *average cost function*, $\bar{c} = c/q$, when $q = 100$ is

$$\bar{c}(100) = (0.05)\cdot 100 + 35 + \frac{12000}{100} = 160.$$

These two quantities are different because they are measuring completely different things, and take different things into account. The value of $\bar{c}$ when $q = 100$ only takes into account the cost of producing 100 units, (and then divides this by 100). The *average of the cost function* on the interval $[0, 100]$, on the other hand, takes all of the values of the cost function on $[0, 100]$ into account, and this give much more weight to the *fixed cost*. Geometrically, the average value of the cost function can be thought of as the average height of the graph of the cost function, and since this height is always bigger than 12000 (why?), the average height will also be bigger than 12000.

6. Consider the sum

$$\sum_{k=1}^{500} 0.4k \cdot e^{-0.0019k}. \tag{1}$$

a. The formula from problem **7.** in the exercises of SN 1 reads

$$\sum_{k=1}^{n} kq^k = \frac{nq^{n+1}}{q-1} - \frac{q^{n+1}-q}{(q-1)^2}.$$

Set $q = e^{-0.0019}$, and recall that $e^{-0.0019k} = (e^{-0.0019})^k = q^k$, so

$$\sum_{k=1}^{500} 0.4k \cdot e^{-0.0019k} = 0.4\left(\frac{500e^{-0.0019\cdot 501}}{e^{-0.0019}-1} - \frac{e^{-0.0019\cdot 501} - e^{-0.0019}}{(e^{-0.0019}-1)^2}\right) \approx 27280.193$$

b. The two numbers differ by about 38.6, which is not very much relative to their sizes.

c. By *definition* of the definite integral

$$\int_0^{20} 250te^{-0.0475t}\,dt = \lim_{n\to\infty}\sum_{k=1}^{n} 250t_k e^{-0.0475t_k}\Delta t_k, \tag{2}$$

where $\Delta t_k = \dfrac{20}{n}$ and $t_k = \dfrac{20k}{n}$.

The sum in (1) is equal to the sum on the right-hand side of (2) for $n = 500$, so its not surprising that the value of the definite integral is somewhat close to the value of this sum.