Review Questions 5

Solutions

1. Compute the following integrals

a. $\displaystyle\int 5x\sqrt{7-2x}\,dx = 5\cdot\int x\sqrt{7-2x}\,dx$

$$= \frac{10(-6x-14)(7-2x)^{3/2}}{60} + C = \boxed{-\frac{(3x+7)(7-2x)^{3/2}}{3} + C}.$$

Use the formula $\displaystyle\int u\sqrt{a+bu}\,du = \frac{2(3bu-2a)(a+bu)^{3/2}}{15b^2} + C$, with $a=7$ and $b=-2$.

b. $\displaystyle\int \frac{7t^2+3t-1}{2+5t}\,dt = 7\cdot\int \frac{t^2\,dt}{2+5t} + 3\cdot\int \frac{t\,dt}{2+5t} - \int \frac{dt}{2+5t}$

$$= 7\left(\frac{t^2}{10} - \frac{2t}{25} + \frac{4}{125}\ln|2+5t|\right) + 3\left(\frac{t}{5} - \frac{2}{25}\ln|2+5t|\right) - \left(\frac{1}{5}\ln|2+5t|\right) + C$$

$$= \boxed{\frac{7}{10}t^2 + \frac{1}{25}t - \frac{27}{125}\ln|2+5t| + C}.$$

Use the formulas (i) $\displaystyle\int \frac{du}{a+bu} = \frac{1}{b}\ln|a+bu| + C$, (ii) $\displaystyle\int \frac{u\,du}{a+bu} = \frac{u}{b} - \frac{a}{b^2}\ln|a+bu| + C$, and

(iii) $\displaystyle\int \frac{u^2\,du}{a+bu} = \frac{u^2}{2b} - \frac{au}{b^2} + \frac{a^2}{b^3}\ln|a+bu| + C$,

(**all** with $a=2$ and $b=5$).

c. $\displaystyle\int 500t^2e^{-0.04t}\,dt = 500\left(\frac{t^2e^{-0.04t}}{-0.04} + \frac{2}{0.04}\int te^{-0.04t}\,dt\right)$

$$= 500\left(\frac{t^2e^{-0.04t}}{-0.04} + \frac{2}{0.04}\left(\frac{e^{-0.04t}}{0.0016}(-0.04t-1)\right)\right) + C$$

$$= \boxed{-12500e^{-0.04t}\left(t^2+50t+1250\right) + C}.$$

Use the formulas (i) $\displaystyle\int u^n e^{au}\,du = \frac{u^n e^{au}}{a} - \frac{n}{a}\int u^{n-1}e^{au}\,du$, (with $n=2$ and $a=-0.04$), and

(ii) $\displaystyle\int ue^{au}\,du = \frac{e^{au}}{a^2}(au-1) + C$, (with $a=-0.04$).

d. $\displaystyle\int \frac{3e^{2x}}{\sqrt{4+e^x}}\,dx = 3\int \frac{e^x}{\sqrt{4+e^x}}\cdot e^x\,dx$

$$= 3\int \frac{u}{\sqrt{4+u}}\,du, \quad \text{(substituting } u=e^x \text{ and } du = e^x\,dx\text{)},$$

$$= 3\left(\frac{2(u-8)\sqrt{4+u}}{3} + C\right) = \boxed{2(e^x-8)\sqrt{4+e^x} + C}.$$

Use the formula $\displaystyle\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2(bu-2a)\sqrt{a+bu}}{3b^2} + C$, (with $a=4$ and $b=1$).

e. $\displaystyle \int \frac{300}{1+0.25e^{-0.1t}}\, dt = 300\frac{-0.01t - \ln|1 + 0.25e^{-0.01t}|}{-0.01} + C$

$$= \boxed{30000\left(0.01t + \ln|1 + 0.25e^{-0.01t}|\right) + C}.$$

Use the formula $\displaystyle \int \frac{du}{a + be^{ku}} = \frac{ku - \ln|a + be^{ku}|}{ak} + C$, (with $a = 1$, $b = 0.25$ and $k = -0.1$).

f. $\displaystyle \int \frac{4(\ln x)^2}{3x\sqrt{2 + 7\ln x}}\, dx = \frac{4}{3}\int \frac{(\ln x)^2}{x\sqrt{2 + 7\ln x}}\, dx$

$$= \frac{4}{3}\int \frac{u^2\, du}{\sqrt{2 + 7u}}, \quad \text{(substituting } u = \ln x,\ du = \tfrac{1}{x}\, dx),$$

$$= \frac{4}{3}\left(\frac{2(147u^2 - 56u + 32)\sqrt{2 + 7u}}{5145} + C\right)$$

$$= \boxed{\frac{8(147(\ln x)^2 - 56\ln x + 32)\sqrt{2 + 7\ln x}}{15435} + C}.$$

Use the formula $\displaystyle \int \frac{u^2\, du}{\sqrt{a + bu}} = \frac{2(3b^2u^2 - 4abu + 8a^2)\sqrt{a + bu}}{15b^3} + C$, (with $a = 2$ and $b = 7$).

2. Let $y = f(x)$ satisfy (i) $\displaystyle \frac{dy}{dx} = 3xy^2$ and (ii) $y(1) = 2$. Find the function $f(x)$.

Separate: $\displaystyle \frac{dy}{y^2} = 3x\, dx$.

Integrate: $\displaystyle \int \frac{dy}{y^2} = \int 3x\, dx \quad \Longrightarrow \quad -\frac{1}{y} = \frac{3x^2}{2} + C$. (This is the *implicit* solution.)

Solve for y: $\displaystyle y = \frac{2}{C - 3x^2}$.

Solve for C: $y(1) = 2 \implies 2 = \dfrac{2}{C - 3} \implies C - 3 = 1 \implies C = 4$.

Solution: $\displaystyle y = \frac{2}{4 - 3x^2}$.

3. The income-elasticity of demand for a firm's product is *proportional* to the square root of income. Find the demand as a function of income, given that $q(100) = 50$ and $q(400) = 90$.

We are told that $\eta_{q/Y} = k\sqrt{Y}$, where k is the (unknown) constant of proportionality. Expressing the elasticity in terms of dq/dY, q and Y gives the differential equation

$$\eta_{q/Y} = \frac{dq}{dY} \cdot \frac{Y}{q} = k\sqrt{Y},$$

which is separable.

Separate: $\displaystyle \frac{dq}{q} = k \cdot \frac{dY}{Y^{1/2}}$.

Integrate: $\int \dfrac{dq}{q} = \int k \cdot Y^{-1/2} \, dY \implies \ln q = 2kY^{1/2} + C = kY^{1/2} + C.$

Notes: (i) I dropped the absolute value from q (in $\ln q$) because $q > 0$ since q is output. (ii) The factor of 2 was absorbed by the unknown constant k.

Solve for q: Exponentiation gives $q = e^{k\sqrt{Y}+C} = Ae^{k\sqrt{Y}}$, where $A = e^C > 0$.

Solve for the parameters, k and A: This is where we use the data $q(100) = 50$ and $q(400) = 90$, leading to the pair of equations

$$
\begin{aligned}
50 &= Ae^{10k} \\
90 &= Ae^{20k}
\end{aligned}
$$

Dividing 90 by 50 on the left and Ae^{20k} by Ae^{10k} on the right, give the equation

$$1.8 = e^{10k} \implies 10k = \ln 1.8 \implies \boxed{k = \dfrac{\ln 1.8}{10} \approx 0.05877...}$$

and pugging $k = (\ln 1.8)/10$ (**not** the approximate value!) back into the first equation gives

$$50 = Ae^{\ln 1.8} = 1.8A \implies \boxed{A = \dfrac{250}{9}}.$$

Thus the demand function is given by

$$q = \dfrac{250}{9} e^{(\ln 1.8)\sqrt{Y}/10}.$$

4. The population of a tropical island grows at a rate that is proportional to the *third root* ($\sqrt[3]{}$) of its size. In 1950, the islands population was 1728 and in 1980, the islands population was 2744. What will the Islands population be in 2020?

First, translate the description of the growth rate into a differential equation. If $P(t) =$ the size of the population at time t (in years), then the description above leads to the differential equation
$$\dfrac{dP}{dt} = k\sqrt[3]{P},$$
where k is the (unknown) constant of proportionality.

Separate the variables: $\dfrac{dP}{P^{1/3}} = k\, dt.$

Integrate both sides: $\int P^{-1/3} \, dP = \int k\, dt \implies \dfrac{3}{2}P^{2/3} = kt + C.$

Solve for P: Multiplying by $2/3$ gives $P^{2/3} = kt + C$, because the factor of $2/3$ is absorbed by both k and C. Next, raise both sides to the power $3/2$ to see that

$$P = (kt + C)^{3/2}.$$

Solve for the parameters k and C: First set $t = 0$ for the year 1950, so

$$1728 = P(0) = (0 + C)^{3/2} = C^{3/2} \quad \Longrightarrow \quad \boxed{C = 1728^{2/3} = 144}.$$

Next, the year 1980 corresponds to $t = 30$, so

$$2744 = P(3) = (30k + 144)^{3/2} \quad \Longrightarrow \quad 30k + 144 = 2744^{2/3} = 196 \quad \Longrightarrow \quad \boxed{k = \frac{26}{15}}.$$

Thus $P(t) = \left(\frac{26}{15}t + 144\right)^{3/2}$ and in 2020 the population will be

$$P(70) = \left(\frac{26}{15} \cdot 70 + 144\right)^{3/2} \approx 4322.$$

5. The population of bass in a large lake grows according to the (logistic) model,

$$\frac{dY}{dt} = 0.05Y(10 - Y),$$

where $Y(t)$ is the size of the bass population, measured in **thousands** of fish.

(a) If the bass population in 1990 was 1500 fish, what will the population be in 2010?

We solved the logistic equation $\frac{dY}{dt} = kY(M - Y)$ in general, and obtained the solution

$$Y = \frac{M}{1 + be^{-kMt}},$$

where b comes from the constant of integration. We can use that knowledge, and conclude that the solution here is given by

$$Y = \frac{10}{1 + be^{-0.5t}},$$

since $M = 10$ and $k = 0.05$ in this example. Next, use the initial data $Y(0) = 1.5$ (remember, Y is measured in 1000s) to solve for b:

$$1.5 = \frac{10}{1 + b} \quad \Longrightarrow \quad 1 + b = \frac{10}{1.5} \quad \Longrightarrow \quad b = \frac{17}{3},$$

so the precise solution here is

$$Y = \frac{10}{1 + \frac{17}{3}e^{-0.5t}}.$$

Now, we can find the population in 2010, by computing $Y(20)$:

$$Y(20) = \frac{10}{1 + \frac{17}{3}e^{-10}} \approx 9.99742.$$

I.e., in 2010 the bass population will be about 9974 fish. This means that by 2010, the bass population will be very close to the carrying capacity of the lake.

(b) When will/did the bass population reach 5000?

Using the solution we found in (a), we solve the equation $Y(t) = 5$ for the variable t:

$$5 = \frac{10}{1 + \frac{17}{3}e^{-0.5t}} \implies 1 + \frac{17}{3}e^{-0.5t} = \frac{10}{5} \implies \frac{17}{3}e^{-0.5t} = 1 \implies e^{-0.5t} = \frac{3}{17}.$$

Taking natural logs of both sides of the equation on the right gives

$$-0.5t = \ln\frac{3}{17} = -\ln\frac{17}{3} \implies t = 2\ln\frac{17}{3} \approx 3.4692.$$

In other words, the bass population reached 5000 a little less than three and half years after the initial population count was done. If the initial data was from January 1, 1990, then the bass population reached the 5000 fish mark on about June 20th, 1993.[†]

(c) Once the population reaches 3000, bass are 'harvested' from the lake at the constant rate of 1000 fish per year. Describe what will happen to the fish population over time.

If the lake is opened to bass fishing (harvesting), then the rate at which the population grows (or shrinks) changes. If Y_H denotes the size of the bass population subject to harvesting, then

$$\frac{dY_H}{dt} = 0.05Y_H(10 - Y_H) - \frac{dH}{dt} = 0.05Y_H(10 - Y_H) - 1 = -0.05Y^2 + 0.5Y - 1,$$

because the harvesting rate (dH/dt) is constant and equal to 1000 fish per year, and the population is measured in 1000s, so $dH/dt = 1$.
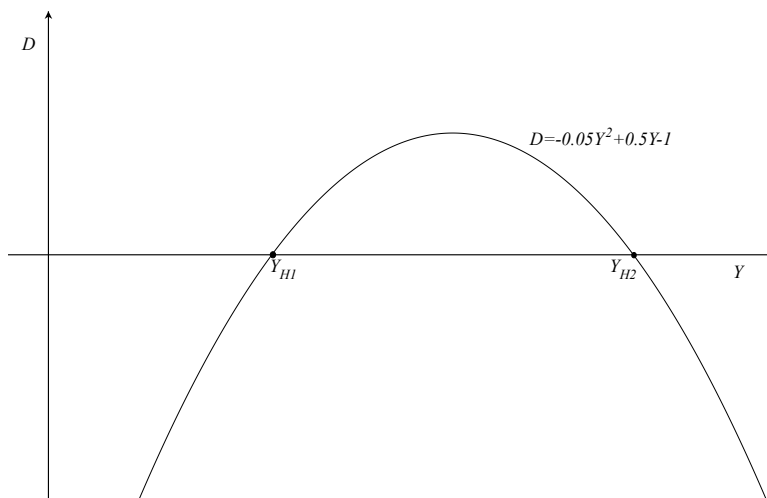

Figure 1: Graph of $D(Y) = -0.05Y^2 + 0.5Y - 1$.

To describe what will happen to the bass population over time, we need to know when the population is increasing or decreasing under the new conditions, and that means we need to figure out the values of Y for which the quadratic function

$$D(Y) = -0.05Y^2 + 0.5Y - 1$$

is positive and for which values of Y it is negative. The graph of this function is depicted in Figure 1. To find where $D > 0$ and where $D < 0$, we need to find the solutions Y_{H1} and Y_{H2} of the quadratic equation $-0.05Y^2 + 0.5Y - 1 = 0$, which we do using the quadratic formula:

$$Y_{H1} = \frac{-0.5 + \sqrt{0.25 - 0.2}}{-0.1} \approx 2.764 \quad \text{and} \quad Y_{H2} = \frac{-0.5 - \sqrt{0.25 - 0.2}}{-0.1} \approx 7.236.$$

[†]This degree of accuracy is unrealistic because the population of fish in a lake does not grow in a nice continuous manner, but rather in discrete jumps.

If $Y_{H1} < Y < Y_{H2}$, then $\dfrac{dY}{dt} = D(Y) > 0$. I.e., if the size of the population when harvesting begins is between 2764 and 7236 fish, then the population will grow larger, but the rate of growth will decrease as the size of the population approaches 7236, and the population will eventually stabilize at this level. Since $2764 < 3000 < 7236$, if nothing else changes, we can expect the bass population to eventually stabilize at approximately 7236. The graph of the fish population (subject to harvesting) as a function of time is depicted in Figure 2.
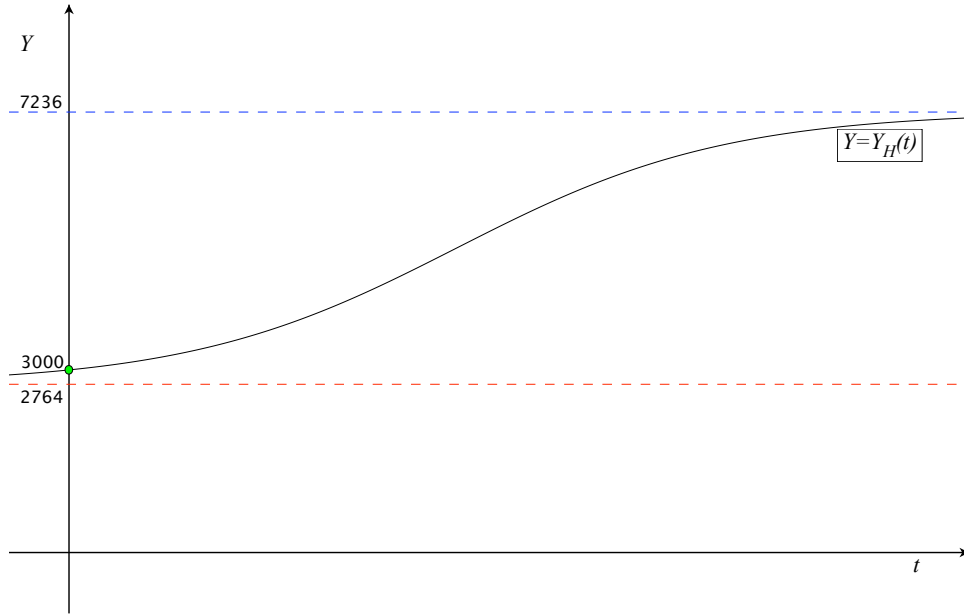

Figure 2: Population of bass, subject to harvesting, as a function of time.