Review Questions 6

Solutions

1. Compute the indicated partial derivatives of the functions below.

a. $z = 3x^2 + 4xy - 5y^2 - 4x + 7y - 2,$

$z_x = 6x + 4y - 4$

$z_y = 4x - 10y + 7$

b. $F(u, v, w) = 60u^{2/3}v^{1/6}w^{1/2}$

$\frac{\partial F}{\partial u} = 40u^{-1/3}v^{1/6}w^{1/2}$

$\frac{\partial^2 F}{\partial w \partial u} = 20u^{-1/3}v^{1/6}w^{-1/2}$

c. $w = x^2 z \ln(y^2 + z^3)$

$w_x = 2xz \ln(y^2 + z^3)$

$w_y = \frac{2x^2yz}{y^2 + z^3}$

$w_{xx} = 2z \ln(y^2 + z^3)$

$w_{yz} = \frac{2x^2y(y^2 + z^3) - 6x^2yz^3}{(y^2 + z^3)^2} = \frac{2x^2(y^3 - 2yz^3)}{(y^2 + z^3)^2}.$

$w_{xyz} = \frac{4x(y^3 - 2yz^3)}{(y^2 + z^3)^2}$, since $w_{xyz} = w_{yzx}$.

d. $f(x, y, z) = 2x^3yz^2 - 3xy^3z + 5x^2y^2 - 7yz^5 + 11x - 1;$

$f_x = 6x^2yz^2 - 3y^3z + 10xy^2 + 11.$

$f_y = 2x^3z^2 - 9xy^2z + 10x^2y - 7z^5.$

$f_{zx} = f_{xz} = 12x^2yz - 3y^3.$

e. $q(u, v) = \frac{u^2v - 3uv^3}{2u + 3v};$

$\frac{\partial q}{\partial u} = \frac{(2uv - 3v^3)(2u + 3v) - 2(u^2v - 3uv^3)}{(2u + 3v)^2} = \frac{2u^2v + 6uv^2 - 9v^4}{(2u + 3v)^2}.$

$\frac{\partial q}{\partial v} = \frac{(u^2 - 9uv^2)(2u + 3v) - 3(u^2v - 3uv^3)}{(2u + 3v)^2} = \frac{2u^3 - 18u^2v^2 - 18uv^3}{(2u + 3v)^2}.$

2. a. $C_{Q_A} = 0.04Q_A + 0.01Q_B + 35$ and $C_{Q_B} = 0.01Q_A + 0.06Q_B + 28$, so $C_{Q_A}(250, 360) = 48.6$ and $C_{Q_B}(250, 360) = 52.1$.

b. Approximation formula: $\Delta C \approx \left(\frac{\partial C}{\partial Q_B} \bigg|_{\substack{Q_A=250 \\ Q_B=360}} \right) \cdot \Delta Q_B$, since we are assuming in this case that $\Delta Q_A = 0$. Now, $\Delta Q_B = 50/100 = 0.5$, so $\Delta C \approx (52.1)(0.5) = 26.05$.

c. General Approximation formula: $\Delta C \approx \left(\left. \frac{\partial C}{\partial Q_A} \right|_{\substack{Q_A=250 \\ Q_B=360}} \right) \cdot \Delta Q_A + \left(\left. \frac{\partial C}{\partial Q_B} \right|_{\substack{Q_A=250 \\ Q_B=360}} \right) \cdot \Delta Q_B$. In this case we have $\Delta Q_A = 60/100 = 0.6$ and $\Delta Q_B = 40/100 = 0.4$, so

$$\Delta C \approx \left(\left. \frac{\partial C}{\partial Q_A} \right|_{\substack{Q_A=250 \\ Q_B=360}} \right) \cdot \Delta Q_A + \left(\left. \frac{\partial C}{\partial Q_B} \right|_{\substack{Q_A=250 \\ Q_B=360}} \right) \cdot \Delta Q_B = (48.6)(0.6) + (52.1)(0.4) = 50.$$

3. a. $Q_K = 0.8K^{-2/3}L^{3/5} + 1.5K^{-1/2}L^{1/4}$ and $Q_L = 1.44K^{1/3}L^{-2/5} + 0.75K^{1/2}L^{-3/4}$, so

$Q_K(200, 320) \approx 1.1936$ and $Q_L(200, 320) \approx 0.9783$.

b. Recall that

$$\eta_{Q/K} = Q_K \cdot \frac{K}{Q} \quad \text{and} \quad \eta_{Q/L} = Q_L \cdot \frac{L}{Q}.$$

Now, we have $K = 200$ and $L = 320$, so $Q \approx 626.446$. Plugging these numbers and te partial derivatives that we computed in part a. into the formulas above, we obtain

$$\eta_{Q/K} \approx (1.1936)\frac{200}{626.446} \approx 0.381 \quad \text{and} \quad \eta_{Q/L} \approx (0.9783)\frac{320}{626.446} \approx 0.5.$$

c. General Approximation Formula for percentage change:

$$\%\Delta Q \approx \eta_{Q/K} \cdot (\%\Delta K) + \eta_{Q/L} \cdot (\%\Delta L).$$

In this case, $\Delta K = 0$, so $\%\Delta K = 0$, and

$$\%\Delta L = \frac{325 - 320}{320} \cdot 100\% = 1.5625\%,$$

so

$$\%\Delta Q \approx 0 + (0.5)(1.5625\%) = 0.78125\%.$$

4. a.

$$\begin{aligned} Q_P &= -\frac{Y_d^{1/3}(2P_s + 5)^{1/2}}{(P+1)^2}, \\ Q_{P_s} &= \frac{Y_d^{1/3}(2P_s + 5)^{-1/2}}{P+1}, \\ Q_{Y_d} &= \frac{Y_d^{-2/3}(2P_s + 5)^{1/2}}{3(P+1)}. \end{aligned}$$

b.

$$\eta_{Q/Y_d} = Q_{Y_d} \cdot \frac{Y_d}{Q} = \frac{Y_d^{-2/3}(2P_s + 5)^{1/2}}{3(P+1)} \cdot \frac{Y_d(P+1)}{Y_d^{1/3}(2P_s + 5)^{1/2}} = \frac{1}{3}.$$

Notice that in this case, the income elasticity is *constant*, so we don't have to worry about plugging in the given values of the variables.

c.

$$\eta_{Q/P} = Q_P \cdot \frac{P}{Q} = -\frac{Y_d^{1/3}(2P_s+5)^{1/2}}{(P+1)^2} \cdot \frac{P(P+1)}{Y_d^{1/3}(2P_s+5)^{1/2}} = -\frac{P}{P+1},$$

so

$$\eta_{Q/P}\Big|_{\substack{P=5.25\\P_s=4.95\\Y_d=2800}} = -\frac{5.25}{6.25} = -0.84.$$

d. To answer this, we use the *approximation formula*

$$\%\Delta Q \approx \eta_{Q/P} \cdot \%\Delta P + \eta_{Q/P_s} \cdot \%\Delta P_s + \eta_{Q/Y_d} \cdot \%\Delta Y_d.$$

By assumption, $\Delta P_s = 0$ (no change), so $\%\Delta P_s = 0$. Also, we are given that $\%\Delta Y_d = 5\%$, but we have to compute $\%\Delta P$:

$$\%\Delta P = \frac{\Delta P}{P} \cdot 100\% = \frac{0.05}{5.25} \cdot 100\% \approx 0.95238\%.$$

Inserting these numbers, together with the elasticities from b. and c., we find that

$$\%\Delta Q \approx -(0.84)(\%0.95238) + (1/3)(5\%) \approx 0.8667\%.$$

5. $\displaystyle \frac{dz}{dt}\Big|_{t=0} = \left(\frac{\partial z}{\partial x}\Big|_{t=0}\right)\left(\frac{dx}{dt}\Big|_{t=0}\right) + \left(\frac{\partial z}{\partial y}\Big|_{t=0}\right)\left(\frac{dy}{dt}\Big|_{t=0}\right) =$

$\left((6xy^3 + 5y^2 - 3)\big|_{t=0}\right) \cdot 3 + \left((9x^2y^2 + 10xy + 1)\big|_{t=0}\right) \cdot 2 =$

$(-12 + 5 - 3) \cdot 3 + (36 - 20 + 1) \cdot 2 = -30 + 34 = 4.$

Note: $x(0) = -2$ and $y(0) = 1$.

6.

$$\begin{aligned}
\frac{\partial P}{\partial m} &= p_0 \frac{\partial Q}{\partial m} - w_0 h \\
&= p_0 \left(\frac{\partial Q}{\partial K} \cdot \frac{\partial K}{\partial m} + \frac{\partial Q}{\partial L} \cdot \frac{\partial L}{\partial m} \right) - w_0 h && \text{(chain rule)} \\
&= p_0 \frac{\partial Q}{\partial L} \cdot \frac{\partial L}{\partial m} - w_0 h && \text{(since } \partial K/\partial m = 0\text{)} \\
&= \boxed{2p_0(60h - h^2)F_L(K,L) - w_0 h} && \text{(remember: } \partial Q/\partial L = F_L\text{)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial P}{\partial h} &= p_0 \frac{\partial Q}{\partial h} - w_0 m \\
&= p_0 \left(\frac{\partial Q}{\partial K} \cdot \frac{\partial K}{\partial h} + \frac{\partial Q}{\partial L} \cdot \frac{\partial L}{\partial h} \right) - w_0 m && \text{(chain rule)} \\
&= p_0 \frac{\partial Q}{\partial L} \cdot \frac{\partial L}{\partial h} - w_0 m && \text{(since } \partial K/\partial h = 0\text{)} \\
&= \boxed{2mp_0(60 - 2h)F_L(K,L) - w_0 m} && \text{(remember: } \partial Q/\partial L = F_L\text{)}
\end{aligned}$$