Review Questions 7

Solutions

1. (a) $\boxed{f(10,7) = (64)^{1/2} = 8}$;

$f_x = \frac{5}{2}(5x+2y)^{-1/2} \implies \boxed{f_x(10,7) = \frac{5}{16}}$;

$f_y = (5x+2y)^{-1/2} \implies \boxed{f_y(10,7) = \frac{1}{8}}$;

$f_{xx} = -\frac{25}{4}(5x+2y)^{-3/2} \implies \boxed{f_{xx}(10,7) = -\frac{25}{2048}}$;

$f_{xy} = -\frac{5}{2}(5x+2y)^{-3/2} \implies \boxed{f_{xy}(10,7) = -\frac{5}{1024}}$;

$f_{yy} = -(5x+2y)^{-3/2} \implies \boxed{f_{yy}(10,7) = -\frac{1}{512}}$;

The quadratic Taylor polynomial for $f(x,y) = \sqrt{5x+2y}$ centered at $(10,7)$:

$$T_2(x,y) = 8+\frac{5}{16}(x-10)+\frac{1}{8}(y-7)-\frac{25}{4096}(x-10)^2-\frac{5}{1024}(x-10)(y-7)-\frac{1}{1024}(y-7)^2.$$

(b) $\boxed{g(0.5,0.25) = \ln(1) = 0}$

$g_u(u,v) = \frac{2u}{u^2+3v} \implies \boxed{g_u(0.5,0.25) = \frac{1}{1} = 1}$

$g_v(u,v) = \frac{3}{u^2+3v} \implies \boxed{g_v(0.5,0.25) = \frac{3}{1} = 3}$

$g_{uu}(u,v) = \frac{6v-2u^2}{(u^2+3v)^2} \implies \boxed{g_{uu}(0.5,0.25) = \frac{1}{1} = 1}$

$g_{uv}(u,v) = \frac{-6u}{(u^2+3v)^2} \implies \boxed{g_{uv}(0.5,0.25) = \frac{-3}{1} = -3}$

$g_{vv}(u,v) = \frac{-9}{(u^2+3v)^2} \implies \boxed{g_{vv}(0.5,0.25) = \frac{-9}{1} = -9}$

The quadratic Taylor polynomial for $g(u,v) = \ln(u^2+3)$ centered at $(0.5, 0.25)$:

$$T_2(u,v) = (u-0.5)+3(v-0.25)+\frac{1}{2}(u-0.5)^2-3(u-0.5)(v-0.25)-4.5(v-0.25)^2.$$

2. Find the critical points of the functions below.

a. $f(x,y) = 3x^2 - 12xy + 19y^2 - 2x - 4y + 5$

First order conditions:

$$\left.\begin{array}{rcccl} f_x & = & 6x - 12y - 2 & = & 0 \\ f_y & = & -12x + 38y - 4 & = & 0 \end{array}\right\}$$

Now, $f_x = 0 \Longrightarrow 6x = 12y + 2$, so $12x = 24y + 4$. Plugging this into the second equation gives

$$\Longrightarrow\ -(24y+4) + 38y - 4 = 0 \Longrightarrow 14y - 8 = 0 \Longrightarrow y_0 = \frac{4}{7} \Longrightarrow x_0 = \frac{31}{21}.$$

So there is one critical point, $\left(\frac{31}{21}, \frac{4}{7}\right)$.

b. $g(s,t) = s^3 + 3t^2 + 12st + 2$

First order conditions:

$$\left.\begin{array}{rcccl} g_s & = & 3s^2 + 12t & = & 0 \\ g_t & = & 6t + 12s & = & 0 \end{array}\right\}$$

Now, $g_t = 0 \Longrightarrow t = -2s$, and plugging this into the first equation gives

$$3s^2 - 24s = 0 \Longrightarrow 3s(s-8) = 0 \Longrightarrow \text{two solutions: } s_1 = 0 \text{ and } s_2 = 8.$$

So there are two critical points in this case, $(s_1, t_1) = (0,0)$ and $(s_2, t_2) = (8, -16)$.

c. $h(u,v) = u^3 + v^3 - 3u^2 - 3v + 5$.

First order conditions:

$$\begin{array}{rcccl} h_u & = & 3u^2 - 6u & = & 0, \\ h_v & = & 3v^2 - 3 & = & 0. \end{array}$$

The first equation factors as $3u(u-2) = 0$, which has two solutions $u_1 = 0$ and $u_2 = 2$. The second equation factors as well, giving $3(v^2 - 1) = 0$, which has the two solutions $v_1 = 1$ and $v_2 = -1$.

The first equation places no restrictions on the variable v, while the second equation places no restrictions on the variable u, so the critical points of the function $h(u,v)$ are

$$(u_1, v_1) = (0,1), \quad (u_1, v_2) = (0,-1), \quad (u_2, v_1) = (2,1) \text{ and } (u_2, v_2) = (2,-1).$$

3. Consider the function $F(x,y;A,B) = 3x^2 - Axy + By^2 - 2x - 4y + 5$, with *variables* x and y and *parameters* A and B.

(a) When $A = A_0 = 12$ and $B = B_0 = 19$, this is the function $f(x,y)$ from 2a., above, and it has one critical point: $(x^*, y^*) = \left(\frac{31}{21}, \frac{4}{7}\right)$. The corresponding critical value is

$$F^* = F(x^*, y^*, A_0, B_0) = f\left(\frac{31}{21}, \frac{4}{7}\right) = \frac{50}{21}.$$

(b) Linear approximation tells us that the change in the critical value, F^* can be approximated by

$$\Delta F^* \approx \frac{\partial F^*}{\partial A} \cdot \Delta A + \frac{\partial F^*}{\partial B} \cdot \Delta B. \tag{1}$$

To compute the partial derivatives $\frac{\partial F^*}{\partial A}$ and $\frac{\partial F^*}{\partial B}$, we use the *envelope theorem*, which says that

$$\left.\frac{\partial F^*}{\partial A}\right|_{\substack{A=A_0\\B=B_0}} = \left.\frac{\partial F}{\partial A}\right|_{\substack{x=x^*\\y=y^*\\A=A_0\\B=B_0}} \quad \text{and} \quad \left.\frac{\partial F^*}{\partial B}\right|_{\substack{A=A_0\\B=B_0}} = \left.\frac{\partial F}{\partial B}\right|_{\substack{x=x^*\\y=y^*\\A=A_0\\B=B_0}}.$$

Now,

$$\frac{\partial F}{\partial A} = -xy \quad \text{and} \quad \frac{\partial F}{\partial B} = y^2,$$

from which it follows that

$$\left.\frac{\partial F^*}{\partial A}\right|_{\substack{A=A_0\\B=B_0}} = \left.\frac{\partial F}{\partial A}\right|_{\substack{x=x^*\\y=y^*\\A=A_0\\B=B_0}} = -x^*y^* = -\frac{124}{147}$$

and

$$\left.\frac{\partial F^*}{\partial B}\right|_{\substack{A=A_0\\B=B_0}} = \left.\frac{\partial F}{\partial B}\right|_{\substack{x=x^*\\y=y^*\\A=A_0\\B=B_0}} = (y^*)^2 = \frac{16}{49}.$$

Finally, plugging these two partial derivatives and the changes $\Delta A = 0.5$ and $\Delta B = 0.2$ in the approximation (1) gives the approximate change in F^*:

$$\Delta F^* \approx -\frac{124}{147} \cdot \frac{1}{2} + \frac{16}{49} \cdot \frac{1}{5} \approx -0.356.$$

4. a. Second derivative test:

$$\left.\begin{array}{rcr} f_{xx} &=& 6 \\ f_{yy} &=& 38 \\ f_{xy} &=& -12 \end{array}\right\} \implies D = 6 \cdot 38 - 144 = 84 > 0.$$

Since $D > 0$ and $f_{xx} > 0$, it follows that $f\left(\frac{31}{21}, \frac{4}{7}\right) = \frac{50}{21}$ is a relative minimum value. (In fact, since the second derivatives are all constant, this is the absolute minimum value.)

b. Second derivative test:

$$\left.\begin{array}{rcr} g_{ss} &=& 6s \\ g_{tt} &=& 6 \\ g_{st} &=& 12 \end{array}\right\} \implies D(s,t) = 36s - 144.$$

Since $D(0,0) = -144 < 0$, the first critical point yields a **saddle point** on the graph of $g(s,t)$ (neither max nor min). Since $D(8,-16) = 144 > 0$ and $g_{ss}(8,-16) = 48 > 0$, it follows that $g(8,-16) = -254$, is a relative minimum value.

Can you show that $g(8,-16) = -254$ is **not** the absolute minimum

c. Second derivative test:

$$\left.\begin{array}{rcl} h_{uu} & = & 6u-6 \\ h_{vv} & = & 6v \\ h_{uv} & = & 0 \end{array}\right\} \Longrightarrow D(u,v) = 36v(u-1).$$

Evaluating the discriminant at the four critical points we find that

i. $D(0,1) = -36 < 0$, so $h(0,1) = 3$ is neither a local minimum value nor a local maximum value;

ii. $D(0,-1) = 36 > 0$ and $h_{uu}(0,-1) = -6 < 0$, so $h(0,-1) = 7$ is a local maximum value;

iii. $D(2,1) = 36 > 0$ and $h_{uu}(2,1) = 6 > 0$, so $h(2,1) = -1$ is a local minimum value; and

iv. $(D(2,-1) = -36$, so $h(2,-1) = 3$ is neither a local minimum value nor a local maximum value.

5. ACME's profit function is

$$\begin{aligned} \Pi &= P_AQ_A + P_BQ_B - C \\ &= 100P_A - 3P_A^2 + 2P_AP_B + 60P_B + 2P_AP_B - 2P_B^2 \\ &\qquad - \left[20(100 - 3P_A + 2P_B) + 30(60 + 2P_A - 2P_B) + 1200\right] \\ &= -3P_A^2 + 4P_AP_B - 2P_B^2 + 100P_A + 80P_B - 5000. \end{aligned}$$

(i) Critical point(s):

$$\left.\begin{array}{rcrcl} \Pi_{P_A} & = & -6P_A + 4P_B + 100 & = & 0 \\ \Pi_{P_B} & = & 4P_A - 4P_B + 80 & = & 0 \end{array}\right\}$$

Now, $\Pi_{P_B} = 0 \Longrightarrow 4P_B = 4P_A + 80$, and plugging this into the first equation gives

$$-6P_A + (4P_A + 80) + 100 = 0 \Longrightarrow -2P_A + 180 = 0 \Longrightarrow P_A = 90 \Longrightarrow P_B = 110.$$

So there is only one critical point $(P_A, P_B) = (90, 110)$.

(ii) Second derivative test:

$$\left.\begin{array}{rcr} \Pi_{P_AP_A} & = & -6 \\ \Pi_{P_BP_B} & = & -4 \\ \Pi_{P_AP_B} & = & 4 \end{array}\right\} \Longrightarrow D = (-6)\cdot(-4) - 16 = 8 > 0.$$

Since $D > 0$ and $\Pi_{P_AP_A} = -6 < 0$, and the second derivatives are all constant, it follows that $\Pi(90, 110) = 3900$ is the absolute maximum profit.

6. Find the critical point(s) of the functions below. You do not need to classify the critical values in this problem.

a. $H(u, v, w) = 2u^2 + v^2 - 3w^2 + 2uv + 4uw - 2vw$.

$$\begin{array}{rcccl} H_u & = & 4u + 2v + 4w & = & 0 \\ H_v & = & 2u + 2v - 2w & = & 0 \\ H_w & = & 4u - 2v - 6w & = & 0. \end{array}$$

The second equation implies that $w = u + v$, and plugging this into the first and third equations gives the pair of equations

$$\begin{array}{rcl} 8u + 6v & = & 0 \\ -2u - 8v & = & 0. \end{array}$$

The second equation in this pair implies that $u = -4v$, and putting this into the first equation of the pair, gives $-26v = 0$, so $v = 0$, which implies that $u = 0$ and $w = 0$. I.e., there is only one critical point, (0,0,0).

b. $F(x, y, z) = 30x^{1/3}y^{2/3} - z(5x + 8y - 400)$.

$$\begin{array}{rcccl} F_x & = & 10x^{-2/3}y^{2/3} - 5z & = & 0, \\ F_y & = & 20x^{1/3}y^{-1/3} - 8z & = & 0, \\ F_z & = & -(5x + 8y - 400) & = & 0. \end{array}$$

Solving the first equation for z gives

$$z = 2x^{-2/3}y^{2/3},$$

and solving the second equation for z gives

$$z = 2.5x^{1/3}y^{-1/3}.$$

This means that

$$2x^{-2/3}y^{2/3} = 2.5x^{1/3}y^{-1/3},$$

since both are equal to z. Multiplying both sides of the last equation by $x^{2/3}y^{1/3}$ gives

$$2y = 2.5x \implies y = 1.25x.$$

Plugging this into the equation $F_z = 0$, gives

$$-5x - 10x + 400 = 0 \implies 15x = 400 \implies x = \frac{80}{3}.$$

This means that $y = \dfrac{100}{3}$ and $z = 2\left(\frac{80}{3}\right)^{-2/3}\left(\frac{100}{3}\right)^{2/3} \approx 2.32$. I.e., there is one critical point: $(80/3, 100/3, 2.32)$.

c. $G(w,x,y,z) = x^2 + 2y^2 + 4z^2 - 2wx - 5wy - 3wz + 300w$.

$$\begin{aligned} G_w &= 300 - 2x - 5y - 3z &= 0, \\ G_x &= 2x - 2w &= 0, \\ G_y &= 4y - 5w &= 0, \\ G_z &= 8z - 3w &= 0. \end{aligned}$$

The second, third and fourth equations imply that

$$w = x, \quad w = \frac{4y}{5} \text{ and } w = \frac{8z}{3},$$

so $x = 4y/5 \implies y = 5x/4$ and $x = 8z/3 \implies z = 3x/8$. Plugging these into the first equation gives

$$300 - 2x - \frac{25x}{4} - \frac{9x}{8} = 0 \implies \frac{75x}{8} = 300 \implies x = 32.$$

So $y = 40$, $z = 12$, $w = 32$ and there is one critical point:

$$(w,x,y,z) = (32, 32, 40, 12).$$

7. The revenue function for this firm is

$$R = P_A Q_A + P_B Q_B = -\frac{3}{2}P_A^2 + 4P_A P_B - 3P_B^2 + 80P_A + 60P_B,$$

which we see by replacing Q_A and Q_B by the rights hand sides of the demand equations. Next, we find the first order partial derivatives and set the equal to 0:

$$\begin{aligned} R_{P_A} = 0 &\implies -3P_A + 4P_B + 80 = 0 \\ R_{P_B} = 0 &\implies 4P_A - 6P_B + 60 = 0. \end{aligned}$$

Solving the second equation for P_B gives

$$P_B = 10 + \frac{2}{3}P_A.$$

Substituting this into the first equation gives

$$-3P_A + 4\left(10 + \frac{2}{3}P_A\right) + 80 = 0 \implies -\frac{P_A}{3} + 120 = 0 \implies P_A = 360.$$

This yields one critical prices $(P_A^*, P_B^*) = (360, 250)$ and the corresponding critical quantities $(Q_A^*, Q_B^*) = (40, 30)$.

Next, we apply the second derivative test:

$$R_{P_A P_A} = -3, \; R_{P_A P_B} = 4 \text{ and } R_{P_B P_B} = -6, \implies D = 18 - 16 = 2 > 0.$$

This implies that the critical value of the revenue

$$R^* = P_A^* Q_A^* + P_B^* Q_B^* = 360 \cdot 40 + 250 \cdot 30 = 21900$$

is a relative maximum value, and since the second derivatives are all constant, it is the absolute maximum revenue.