**Review Questions 8**

# Solutions

***Note:*** In these problems, you may generally assume that the critical point(s) you find produce the required optimal value(s). At the same time, you should see if you can find an argument to justify this assumption in each example.

**1a.** Find the minimum value of $f(x,y) = x^2 + y^2$ subject to $3x + 5y = 68$.

***Lagrangian***: $F(x,y,\lambda) = x^2 + y^2 - \lambda(3x + 5y - 68)$.

***'Structural' equations***:

$$\begin{array}{rcccl} F_x &=& 2x - 3\lambda &=& 0 \\ F_y &=& 2y - 5\lambda &=& 0 \end{array}$$

Solving these equations for $\lambda$ gives

$$\lambda = \frac{2x}{3} = \frac{2y}{5} \implies x = \frac{3y}{5}.$$

Substituting this into the constraint gives

$$3\left(\frac{3y}{5}\right) + 5y = 68 \implies 34y = 340 \implies \boxed{y_0 = 10 \text{ and } x_0 = 6}.$$

The minimum value of $x^2 + y^2$ subject to $3x + 5y = 68$ is therefore obtained at the point $(6, 10)$, giving $6^2 + 10^2 = 136$.

**1b.** Find the maximum value of $g(x,y,z) = 20x^{1/2}y^{1/3}z^{1/6}$, subject to the constraint $5x + 4y + 7z = 1680$.

***Lagrangian:*** $F(x,y,z,\lambda) = 20x^{1/2}y^{1/3}z^{1/6} - \lambda(5x + 4y + 7z - 1680)$.

***'Structural' equations***:

$$\begin{array}{rcccl} F_x &=& 10x^{-1/2}y^{1/3}z^{1/6} - 5\lambda &=& 0 \\ F_y &=& (20/3)x^{1/2}y^{-2/3}z^{1/6} - 4\lambda &=& 0 \\ F_z &=& (10/3)x^{1/2}y^{1/3}z^{-5/6} - 7\lambda &=& 0. \end{array}$$

Solving these equations for $\lambda$ yields the triple equation

$$(\lambda =) \qquad \frac{2y^{1/3}z^{1/6}}{x^{1/2}} = \frac{5x^{1/2}z^{1/6}}{3y^{2/3}} = \frac{10x^{1/2}y^{1/3}}{21z^{5/6}}.$$

Comparing the first and second expressions, canceling the factor of $z^{1/6}$ from both and clearing denominators gives

$$6y = 5x \implies y = \frac{5x}{6}.$$

Comparing the first and third expressions, canceling the factor of $y^{1/3}$ from both and clearing denominators gives

$$42z = 10x \implies z = \frac{5x}{21}.$$

Substituting the expressions for $y$ and $z$ into the constraint gives

$$5x + 4\left(\frac{5x}{6}\right) + 7\left(\frac{5x}{21}\right) = 1680 \implies 420x = 70560 \implies \boxed{x_0 = 168,\ y_0 = 140,\ z_0 = 40.}$$

Thus, the maximum value of $f(x, y, z) = 20x^{1/2}y^{1/3}z^{1/6}$, subject to the constraint $5x+4y+7z = 1680$ is given by $\boxed{f(168, 140, 40) \approx 2489.262}$.

**1c.** Find the maximum and minimum values of the function $h(x, y) = 3x + 5y$ subject to the constraint $x^2 + y^2 = 136$.

***Lagrangian:*** $F(x, y, \lambda) = 3x + 5y - \lambda(x^2 + y^2 - 136)$.

***'Structural' equations***:

$$\begin{aligned} F_x &= 3 - 2x\lambda = 0 \\ F_y &= 5 - 2y\lambda = 0. \end{aligned}$$

Solving these equations for $\lambda$ gives

$$\lambda = \frac{3}{2x} = \frac{5}{2y} \implies 6y = 10x \implies y = \frac{5x}{3}.$$

Substituting this expression for $y$ into the constraint gives

$$x^2 + \left(\frac{5x}{3}\right)^2 = 136 \implies 34x^2 = 1224 \implies x^2 = 36.$$

There are two critical $x$-values, $x_1 = 6$ and $x_2 = -6$, so there are two critical points: $(x_1, y_1) = (6, 10)$ and $(x_2, y_2) = (-6, -10)$. The two critical values are $h(6, 10) = 68$, which is the constrained maximum value, and $h(-6, -10) = -68$, which is the constrained minimum value.

**2.** The objective function is the utility $U(x, y, z) = 5\ln x + 7\ln y + 18\ln z$, and the constraint is the budget (or income) constraint we obtain from the prices and the budget: $xp_x + yp_y + zp_z = \beta \implies 4x + 8y + 30z = 1200$.

**a.** ***Lagrangian:*** $F(x, y, z, \lambda) = 5\ln x + 7\ln y + 18\ln z - \lambda(4x + 8y + 30z - 1200)$.

***'Structural' equations:***

$$\begin{aligned} F_x &= \frac{5}{x} - 4\lambda = 0 \\ F_y &= \frac{7}{y} - 8\lambda = 0 \\ F_z &= \frac{18}{z} - 30\lambda = 0. \end{aligned}$$

Solving these equations for $\lambda$ gives the triple equation

$$\lambda = \frac{5}{4x} = \frac{7}{8y} = \frac{3}{5z}.$$

Comparing the $x$-term and the $y$-term and clearing denominators gives

$$40y = 28x \implies y = \frac{7x}{10}.$$

Comparing the $x$-term and the $z$-term and clearing denominators gives

$$25z = 12x \implies z = \frac{12x}{25}.$$

Substituting the expressions for $y$ and $z$ that we found into the budget constraint gives

$$4x + 8\left(\frac{7x}{10}\right) + 30\left(\frac{12x}{25}\right) = 1200 \implies 1200x = 60000 \implies \boxed{x_0 = 50,\ y_0 = 35,\ z_0 = 24.}$$

Thus, Jack maximizes his utility by consuming 50 fast food meals, 35 diner meals and 24 'fancy' restaurant meals in a month, resulting in a utility of $U(50, 35, 24) \approx 101.652$.

**b.** Since the utility function and the prices of meals are not changing, the maximum possible utility, $U_{\text{max}}$, is a function of the budget, $\beta$. I.e., increasing the budget increases $U_{\text{max}}$ and decreasing the budget decreases $U_{\text{max}}$.

The ***envelope theorem*** tells us that

$$\frac{dU_{\text{max}}}{d\beta} = \lambda_0,$$

where $\lambda_0$ is the critical value of the multiplier $\lambda$. In this case,

$$\lambda_0 = \frac{5}{4x_0} = \frac{5}{200} = 0.025.$$

Hence by the approximation formula,

$$\Delta U_{\text{max}} \approx \lambda_0 \cdot \Delta\beta = 0.025 \cdot 50 = 1.25.$$

In other words, if Jack's food budget increases by \$50.00, then his (maximum possible) utility will increase by approximately 1.25.

**3.** The objective function in this case is the *cost* function

$$C(K, L) = 1280K + 14580L,$$

which is the cost of using $K$ units of capital input and $L$ units of labor input. The constraint in this case is given by the equation

$$10K^{2/5}L^{3/5} = 20480,$$

since the task here is to *minimize the cost* of producing 20480 drills.

**a.** ***Lagrangian:*** $F(K, L, \lambda) = 1280K + 14580L - \lambda(10K^{2/5}L^{3/5} - 20480)$.

***'Structural' equations:***

$$\begin{aligned} F_K &= 1280 - 4\lambda K^{-3/5}L^{3/5} &= 0 \\ F_L &= 14580 - 6\lambda K^{2/5}L^{-2/5} &= 0. \end{aligned}$$

Solving these equations for $\lambda$ gives

$$\lambda = \frac{320K^{3/5}}{L^{3/5}} = \frac{2430L^{2/5}}{K^{2/5}} \implies 320K = 2430L \implies \boxed{K = \frac{243}{32}L}.$$

Plugging this expression for $K$ into the constraint gives the critical $L$ value,

$$10\left(\tfrac{243}{32}L\right)^{2/5} \cdot L^{3/5} = 20480 \implies \frac{90}{4}L = 20480 \implies \boxed{L^* = \frac{2}{45} \cdot 20480 = \frac{8192}{9}},$$

and the critical value capital input is obtained using the relationship boxed above,

$$\boxed{K^* = \frac{243}{32}L^* = 6912}$$

**b.** This is easier than it may appear at first glance, since we have done most of the work already in part **a.** All we need to do is replace 20480 by $q$ in our formula for $L^*$, i.e., in the second boxed equation in part **a.**

If we denote by $L^*(q)$ and $K^*(q)$ the cost minimizing levels of labor and capital input necessary to produce $q$ drills, then

$$\boxed{L^*(q) = \frac{2}{45}q}$$

and

$$\boxed{K^*(q) = \frac{243}{32} \cdot L^*(q) = \frac{27}{80}q.}$$

**c.** Denote by $\lambda^*(q)$ the critical value of the *multiplier* in the optimization problem of minimizing the cost of producing $q$ drills and denote by $C^*(q)$ the minimal cost of producing $q$ drills. Then, by the envelope theorem we may conclude that

$$\frac{dC^*}{dq} = \lambda^*(q) = \frac{2430(L^*(q))^{2/5}}{(K^*(q))^{2/5}} = 2430\left(\frac{\frac{2}{45}q}{\frac{27}{80}q}\right)^{2/5} = 1080.$$

I.e., the drill company's marginal cost is *constant.*

**d.** With a *constant* marginal cost of 1080 per drill, and fixed costs of $c_0 = \$100,000$, the firm's cost function must be

$$c = 1080q + 100000.$$

**4.** To use the envelope theorem, we need to rewrite the firm's profit function in a way that emphasizes the role of the parameter $C_{Q_A}$ in the profit function. The firm's cost function is

$$C = 20Q_A + 30Q_B + 1200,$$

so $C_{Q_A} = 20$. To keep track of this parameter in the profit function, I'll replace 20 by $C_{Q_A}$ in the profit function. Starting with the expression given in the solution to RQ 2, problem 3, we rewrite $\Pi$ (slightly) as follows.

$$\begin{aligned} \Pi &= R(P_A, P_B, Q_A, Q_B) - C(Q_A, Q_B) \\ &= P_A Q_A + P_B Q_B - C_{Q_A} \cdot Q_A - 30Q_B - 1200 \\ &= -3P_A^2 + 4P_A P_B - 2P_B^2 + 40P_A + 120P_B + C_{Q_A}(3P_A - 2P_B - 100) - 3000 \end{aligned}$$

If $C_{Q_A} = 20$, then

$$\Pi = -3P_A^2 + 4P_A P_B - 2P_B^2 + 100P_A + 80P_B - 5000,$$

and we found in RQ 2 that profit is maximized at the critical prices $P_A^* = 90$ and $P_B^* = 110$, with a corresponding maximum profit of $\Pi^* = 3900$. According to the envelope theorem

$$\left.\frac{\partial \Pi^*}{\partial C_{Q_A}}\right|_{C_{Q_A}=20} = \left.\frac{\partial \Pi}{\partial C_{Q_A}}\right|_{\substack{P_A=P_A^* \\ P_B=P_B^* \\ C_{Q_A}=20}} = 3P_A^* - 2P_B^* - 100 = -50.$$

If $C_{Q_A}$ rises from 20 to 20.5, then by linear approximation

$$\Delta\Pi^* \approx \frac{\partial \Pi^*}{\partial C_{Q_A}} \cdot \Delta C_{Q_A} = 50 \cdot 1 = -50.$$

I.e., if the marginal cost (to the firm) of producing product A increases from 20 per unit to 21 per unit, then their (max) profit will decrease by about 50.

**Comment:** If you replace the cost function $C = 20Q_A + 30Q_B + 1200$ by the new cost function $C_1 = 21Q_A + 30Q_B + 1200$ and redo the maximization problem (RQ 2, #3), you will find that the new critical prices are $P_A^* = 90.5$ and $P_B^* = 110$, and the new max profit is $\Pi^* = 3850.75$.[†] There are two interesting phenomena to note. First, the combination of the envelope theorem and linear approximation predicts the change in the firm's profit quite accurately. Second, the firm's optimal reaction to the increase in the marginal cost of product A is to increase the price of product A. The price of product B remains unchanged.

[†] As you can (and should) check for yourself.