Review Questions 9

Solutions

1. a. $Q(20, 15, 5) \approx 575.866$

b. First compute the marginal products of labor and real estate at the given point:

$$\begin{aligned} Q_L = 15K^{2/5}L^{-1/2}R^{1/4} \quad &\Longrightarrow \quad Q_L(20, 15, 5) \approx 19.195, \\ Q_R = 7.5K^{2/5}L^{1/2}R^{-3/4} \quad &\Longrightarrow \quad Q_R(20, 15, 5) \approx 28.793. \end{aligned}$$

Now, use linear approximation:

$$\begin{aligned} \Delta Q &\approx Q_L(20, 15, 5) \cdot \Delta L + Q_R(20, 15, 5) \cdot \Delta R \\ &\approx (19.195)(0.75) + (28.793)(0.5) \approx 28.793. \end{aligned}$$

2. a. We want to maximize the output, $Q = 30K^{2/5}L^{1/2}R^{1/4}$, subject to the budget constraint $K + L + R = 69$, since the inputs are all being measured in \$millions. The Lagrangian for this problem is

$$F(K, L, R, \lambda) = 30K^{2/5}L^{1/2}R^{1/4} - \lambda(K + L + R - 69),$$

and the first order conditions are

$$\begin{aligned} F_K = 0 \quad &\Longrightarrow \quad 12K^{-3/5}L^{1/2}R^{1/4} = \lambda, \\ F_L = 0 \quad &\Longrightarrow \quad 15K^{2/5}L^{-1/2}R^{1/4} = \lambda, \\ F_R = 0 \quad &\Longrightarrow \quad 7.5K^{2/5}L^{1/2}R^{-3/4} = \lambda, \\ F_\lambda = 0 \quad &\Longrightarrow \quad K + L + R = 69. \end{aligned}$$

The first two equations imply that

$$12K^{-3/5}L^{1/2}R^{1/4} = 15K^{2/5}L^{-1/2}R^{1/4} \implies \frac{12L^{1/2}}{K^{3/5}} = \frac{15K^{2/5}}{L^{1/2}},$$

after canceling the common factor of $R^{1/4}$. Clearing denominators gives

$$12L = 15K \implies \boxed{L = 1.25K}.$$

Likewise, comparing the first and third equations implies that

$$12K^{-3/5}L^{1/2}R^{1/4} = 7.5K^{2/5}L^{1/2}R^{-3/4} \implies \frac{12R^{1/4}}{K^{3/5}} = \frac{7.5K^{2/5}}{R^{3/4}},$$

and clearing denominators gives

$$12R = 7.5K \implies \boxed{R = 0.625K}.$$

Substituting for R and L in the fourth equation (the constraint) gives

$$K + 1.25K + 0.625K = 69 \implies 2.875K = 69.$$

Thus, the critical values for the inputs are

$$K^* = 24, \quad L^* = 30 \text{ and } R^* = 15,$$

and the hotel chain's maximum output is

$$Q^* = Q(24, 30, 15) \approx 1152.894.$$

b. When output is maximized, the critical value of λ is

$$\lambda^* = 12(K^*)^{-3/5}(L^*)^{1/2}(R^*)^{1/4} \approx 19.215.$$

c. By the envelope theorem,

$$\frac{dQ^*}{dB} = \lambda^*,$$

where B is the budget. It follows that

$$\Delta Q^* \approx \lambda^* \cdot \Delta B \approx 9.607,$$

since we measure the budget in the same units as the inputs, so an increase of \$500,000, means that $\Delta B = 0.5$.

3. The equation $\eta_{q/p} = -0.1\sqrt{p}$ gives a ***differential equation*** for the demand function, namely

$$\frac{dq}{dp} \cdot \frac{p}{q} = -0.1p^{1/2}.$$

First, separate the variables:

$$\frac{dq}{q} = -0.1p^{-1/2}\,dp.$$

Then, integrate both sides:

$$\int \frac{dq}{q} = \boxed{\ln q = -0.2p^{1/2} + C} = \int -0.1p^{-1/2}\,dp.$$

Next, exponentiate to solve for q:

$$q = e^{-0.2p^{1/2}+C} = Ae^{-\sqrt{p}/5},$$

where $A = e^C$. Finally, use the data $q(16) = 200$ to solve for A:

$$200 = Ae^{-4/5} \implies A = 200e^{4/5} \implies q = 200e^{4/5}e^{-\sqrt{p}/5} = 200e^{(4-\sqrt{p})/5},$$

so $q(25) = 200e^{-1/5} \approx 163.75$.

4. The first step is to find the point of market equilibrium. First we find the equilibrium demand, $\tilde{q}$:

$$0.01q^2 + 5 = p = 60 - 0.6q \implies 0.01q^2 + 0.6q - 55 = 0 \implies q = \frac{-0.6 \pm \sqrt{0.36 + 2.2}}{0.02},$$

so $\tilde{q} = 50$, since the other solution ($q = -110$) is negative. The equilibrium price is then $\tilde{p} = 60 - 0.6\tilde{q} = 30$.

Next, consumers' surplus ($CS = \int_0^{\tilde{q}} (demand - \tilde{p})\, dq$) ...

$$CS = \int_0^{50} (60 - 0.6q) - 30\, dq = \left. 30q - 0.3q^2 \right|_0^{50} = 750,$$

... and producers' surplus ($PS = \int_0^{\tilde{q}} (\tilde{p} - supply)\, dq$)

$$PS = \int_0^{50} 30 - (0.01q^2 + 5)\, dq = \left. 25q - \frac{0.01}{3} q^3 \right|_0^{50} = \frac{2500}{3}.$$

5. a. We need to optimize the utility function $U = 3\ln(x+2) + 2\ln(y+3) + \ln(z+5)$, subject to the constraint $10x + 20y + 25z = 1500$. The Lagrangian for this problem is

$$F(x, y, z, \lambda) = 3\ln(x+2) + 2\ln(y+3) + \ln(z+5) - \lambda(10x + 20y + 25z - 1500),$$

and the first order conditions for (constrained) optimization are

$$\begin{array}{lcl} F_x = 0 & \Longrightarrow & \dfrac{3}{x+2} = 10\lambda, \\ F_y = 0 & \Longrightarrow & \dfrac{2}{y+3} = 20\lambda, \\ F_z = 0 & \Longrightarrow & \dfrac{1}{z+5} = 25\lambda, \\ F_\lambda = 9 & \Longrightarrow & 10x + 20y + 25z = 1500. \end{array}$$

Solving the first three equations for λ gives

$$\begin{aligned} \lambda &= \frac{3}{10(x+2)} \\ \lambda &= \frac{1}{10(y+3)} \\ \lambda &= \frac{1}{25(z+5)}. \end{aligned}$$

Setting the first and second expressions for λ equal to each other, clearing denominators and simplifying gives

$$3(y+3) = (x+2) \Longrightarrow \boxed{y = \frac{x}{3} - \frac{7}{3}.}$$

Setting the first and third expressions for λ equal to each other and multiplying both sides by $(x+2)(z+5)$ gives

$$0.3(z+5) = 0.04(x+2) \Longrightarrow \boxed{z = \frac{2x}{15} - \frac{71}{15}.}$$

Substituting these expressions for y and z into the constraint gives

$$10x + 20\overbrace{\left(\frac{x}{3} - \frac{7}{3}\right)}^{y} + 25\overbrace{\left(\frac{2x}{15} - \frac{71}{15}\right)}^{z} = 1500 \Longrightarrow \boxed{20x = 1665.}$$

Thus, the critical values for x, y and z are

$$x^* = 83.25, \quad y^* = \frac{305}{12} \approx 25.42 \text{ and } z^* = \frac{191}{30} \approx 6.37,$$

and the maximum utility is

$$U^* = U(x^*, y^*, z^*) \approx 22.462.$$

b. By linear approximation,

$$\Delta U^* \approx \frac{dU^*}{dB} \cdot \Delta B.$$

If the consumer wants to increase her max utility by one unit, then $\Delta U^* = 1$ and the approximation $1 \approx \frac{dU^*}{dB} \cdot \Delta B$ implies that the change in budget should be

$$\Delta B \approx \frac{1}{dU^*/dB}.$$

By the envelope theorem,

$$\frac{dU^*}{dB} = \lambda^* = \frac{3}{10(x^*+2)} = \frac{3}{852.5} \approx 0.00352,$$

so

$$\Delta B \approx 1/\lambda^* \approx 284.17.$$

I.e., to increase her utility by one *util*, the consumer will have to increase her budget by about \$284.17.

6. The present value is given by the integral $\int_0^{10} 200t^2 e^{-0.045t}\,dt$, which we compute using formulas #39 ***and*** #38 from appendix C in the book:

$$\begin{aligned}
\int_0^{10} 200t^2 e^{-0.045t}\,dt &= \left.\frac{200t^2 e^{-0.045t}}{-0.045}\right|_0^{10} - \frac{2}{-0.045}\int_0^{10} 200te^{-0.045t}\,dt \\
&= -283390.29 + \frac{400}{0.045}\left[\left.\frac{e^{-0.045t}}{(0.045)^2}(-0.045t-1)\right|_0^{10}\right] \\
&= -283390.29 + \frac{400}{(0.045)^3}\left(e^{-0.45}(-1.45)+1\right) \\
&= -283390.29 + 331145.92 \\
&= 47755.63.
\end{aligned}$$

The first integral was computed using formula #39 ($a = -0.045$ and $n = 2$), and the second integral was computed using formula #38 ($a = -0.045$ again).

7.

a. The firm's total daily output is $Q_A + Q_B$, so the firm's daily cost is

$$C = 40(Q_A + Q_B) + 2500.$$

The firm's revenue from market A is $R_A = P_A Q_A$ and the revenue from market B is $R_B = P_B Q_B$. The firm's profit function is

$$\begin{aligned}
\Pi &= P_A Q_A + P_B Q_B - C \\
&= P_A(100 - 0.4P_A) + P_B(120 - 0.5P_B) - [40(100 - 0.4P_A + 120 - 0.5P_B) + 2500] \\
&= -0.4P_A^2 - 0.5P_B^2 + 116P_A + 140P_B - 11300.
\end{aligned}$$

The first order conditions for an optimum value give

$$\begin{aligned}
\Pi_{P_A} &= 0 \implies -0.8P_A + 116 = 0 \implies \boxed{P_A^* = 145} \\
\Pi_{P_B} &= 0 \implies -P_B + 140 = 0 \implies \boxed{P_B^* = 140}
\end{aligned}$$

The second order conditions for a maximum are

$$\Pi_{P_AP_A}(P_A^*, P_B^*) \cdot \Pi_{P_BP_B}(P_A^*, P_B^*) - (\Pi_{P_AP_B}(P_A^*, P_B^*))^2 > 0 \;\text{ and }\; \Pi_{P_AP_A}(P_A^*, P_B^*) < 0.$$

In this case we have $\Pi_{P_AP_A} = -0.8 < 0$ and

$$\Pi_{P_AP_A}(P_A^*, P_B^*) \cdot \Pi_{P_BP_B}(P_A^*, P_B^*) - (\Pi_{P_AP_B}(P_A^*, P_B^*))^2 = 0.8 > 0$$

for all (P_A, P_B), so that the second order conditions are satisfied, and because the conditions hold everywhere, the critical value of profit

$$\Pi^* = \Pi(P_A^*, P_B^*) = 6910$$

is the absolute maximum daily profit.

<u>To summarize:</u> the firm's profit is maximized when $P_A^* = 145$ and $P_B^* = 140$, at which point the maximum profit is $\Pi^* = 6910$.

b. Returning to the original expression of the profit function and labeling the marginal cost by μ, we have

$$\Pi = P_A(100 - 0.4P_A) + P_B(120 - 0.5P_B) - [\mu(100 - 0.4P_A + 120 - 0.5P_B) + 2500].$$

According to the envelope theorem,

$$\left.\frac{d\Pi^*}{d\mu}\right|_{\mu=40} = \left.\frac{\partial \Pi}{\partial \mu}\right|_{\substack{P_A=P_A^* \\ P_B=P_B^* \\ \mu=40}} = -(100 - 0.4P_A^* + 120 - 0.5P_B^*) = -92.$$

Now, using linear approximation we have

$$\Delta\Pi^* \approx \left.\frac{d\Pi^*}{d\mu}\right|_{\mu=40} \cdot \Delta\mu = -92 \cdot 2 = -184.$$

b. The sales tax does ***not affect*** the firm's ***cost***, but it ***does change*** the firm's ***revenue*** in each market, since it raises the price of the firm's product and so affects the demand.

Specifically, the price with tax to consumers is $1.1P_A$ in market A and $1.1P_B$ in market B, so the new demands in each of these markets will be

$$\widetilde{Q}_A = 100 - 0.4(1.1P_A) = 100 - 0.44P_A \quad \text{and} \quad \widetilde{Q}_B = 120 - 0.5(1.1P_B) = 120 - 0.55P_B,$$

respectively. This means that the firm's new profit function is

$$\widetilde{\Pi} = P_A\widetilde{Q}_A + P_B\widetilde{Q}_B - C(\widetilde{Q}_A + \widetilde{Q}_B) = -0.44P_A^2 - 0.55P_B^2 + 117.6P_A + 142P_B - 11300.$$

The first order conditions are now

$$\begin{aligned} \Pi_{P_A} &= 0 \implies -0.88P_A + 117.6 = 0 \implies \boxed{\widetilde{P}_A^* \approx 133.64} \\ \Pi_{P_B} &= 0 \implies -1.1P_A + 142 = 0 \implies \boxed{\widetilde{P}_B^* \approx 129.09}, \end{aligned}$$

and since the second order conditions for a maximum are still satisfied (check!),

$$\widetilde{\Pi}^* = \widetilde{\Pi}(\widetilde{P}_A^*, \widetilde{P}_B^*) \approx 5723.27$$

is the firm's new maximum profit.

The government's daily revenue from the tax is 10% of the firm's ***revenue*** (not profit — that's taxed elsewhere), i.e.,

$$GR = 0.1\left[\widetilde{P}_A^*(100 - 0.44\widetilde{P}_A^*) + \widetilde{P}_B^*(120 - 0.55\widetilde{P}_B^*)\right] \approx 1183.13.$$

The cost to the firm is the difference between the old maximum profit and the new:

$$\Pi^* - \widetilde{\Pi}^* = 6910 - 5723.27 = 1186.87.$$

The effect on consumers is that the total price per unit has increased. The costs per unit to consumers in each market are now

$$1.1\widetilde{P}_A^* = 147 \quad \text{and} \quad 1.1\widetilde{P}_B^* = 142,$$

respectively, i.e. the cost is now \$2 more per unit to consumers in both markets. This leads to a slight decrease in overall daily consumption of the firm's product of about \$3.74.

8. I use the formula $\gamma = 1 - 2\displaystyle\int_0^1 f(x)\,dx$ that we derived in class for the Gini coefficient, γ, where $y = f(x)$ is the Lorenz curve. You should also look at problem 35 in section 14.10. In this case

$$\gamma = 1 - 2\int_0^1 \frac{10^x - 1}{9}\,dx = 1 - 2\left[\frac{10^x}{9(\ln 10)} - \frac{x}{9}\bigg|_0^1\right] = 1 - 2\left[\frac{10}{9(\ln 10)} - \frac{1}{9} - \frac{1}{9(\ln 10)}\right] \approx 0.3536.$$

9. Call the firm's profit function $\Pi(q)$, then $\Pi(q) = r(q) - c(q)$, and so $\dfrac{d\Pi}{dq} = \dfrac{dr}{dq} - \dfrac{dc}{dq}$. The total change in profit, when output increases from $q = 10$ to $q = 20$, is given by the definite integral

$$\Delta\Pi = \int_{10}^{20} \frac{d\Pi}{dq}\, dq,$$

which in this case may be expressed explicitly as

$$\boxed{\Delta\Pi = \int_{10}^{20} 0.4q\sqrt{400 - 0.64q^2} - (0.2q + 25)\, dq}.$$

To compute the integral, we first split it into two integrals,

$$\Delta\Pi = \int_{10}^{20} 0.4q\sqrt{400 - 0.64q^2}\, dq - \int_{10}^{20} 0.2q + 25\, dq \tag{1}$$

and compute each one separately. To compute the integral on the left, make the substitution $u = 400 - 0.64q^2$, so $du = -1.28q\, dq$ and $0.4q\, dq = -(5/16)\, du$. The limits of integration also change:

$$q = 10 \implies u = 400 - 64 = 336 \quad \text{and} \quad q = 20 \implies u = 400 - 256 = 144.$$

Thus

$$\int_{10}^{20} 0.4q\sqrt{400 - 0.64q^2}\, dq = -\frac{5}{16}\int_{336}^{144} u^{1/2}\, du = -\frac{5}{16} \cdot \frac{u^{3/2}}{3/2}\bigg|_{336}^{144} = -\frac{5}{24}\left(144^{3/2} - 336^{3/2}\right) \approx 923.12.$$

The integral on the right-hand side of (1) is easy:

$$\int_{10}^{20} 0.2q + 25\, dq = \frac{0.2q^2}{2} + 25q\bigg|_{10}^{20} = 40 + 500 - (10 + 250) = 280.$$

Putting everything together, we have

$$\boxed{\Delta\Pi \approx 923.12 - 280 = 643.12.}$$